DOCUMENTS DE TREBALL

DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E09/229

Complements and substitutes in multilateral assignment markets

Oriol Tejada

Adreça correspondència:

Oriol Tejada
Department of Actuarial, Financial and Economic Mathematics
Facultat d'Economia i Empresa
Universitat de Barcelona
Avda. Diagonal, 690.
08034 Barcelona (Spain)
E-mail: oriol.tejada@ub.edu

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Abstract: I prove that, in assignment markets with more than two sides, agents of different sides (or sectors) need not be complements, whereas agents of the same side need not be substitutes. Shapley (1962) showed that this cannot happen when assignment markets are bilateral. Nevertheless, I found sufficient conditions, that always hold for bilateral markets, that guarantee substitutability and a extended notion of complementarity among agents in arbitrary multilateral assignment markets. I also prove that Shapley's (1962) result always holds regardless the number of sectors of the market when goods in the market are homogeneous.

Keywords: Assignment problem, multi-sided assignment markets, complementarity, substitutability, homogeneous goods

JEL Classification: C70, C78

Resum: En aquest treball provo que, en mercats d'assignació amb més de dos costats, agents de diferents sectors poden no ser complementaris mentre que agents del mateix sector poden no ser substituts. Shapley (1962) va provar que això mai pot succeïr quan el mercat d'assignació només té dos costats. No obstant, demostro que existeixen condicions suficients que garanteixen la substitutabilitat i la complementarietat entre agents en aquests tipus de mercats. A més, provo que, quan els béns al mercat son homogenis, el resultat de Shapley (1962) es manté.

Agraïments: I would like to thank Carles Rafels, Francisco Javier Martínez de Albéniz and Marina Nuñez for their comments and suggestions. Support from grant Programa FPU of Ministerio de Educación y Ciencia of the Spanish Government and from the Barcelona Economics Program of CREA, Ministerio de Educación y Ciencia and FEDER, under grant ECO2008-02344/ECON is acknowledged.

1 Introduction

Consider a market with different types (or sectors) of agents, where a worth is attached to any coalition of exactly one agent of each type. These worths account for the benefits that can be generated by any of these *essential coalitions*. The benefit created by an arbitrary coalition is obtained partitioning it into essential coalitions and adding up their worths so that the sum is maximum. This type of markets, that are called *multi-sided assignment markets*, are introduced by Shapley and Shubik (1972) for only two sectors, and generalized by Quint (1991) for an arbitrary number of sides.

Agents can be assigned among them, forming essential coalitions, in a finite number of different ways, being at least one that is best (or optimal), in the sense that it maximizes the aggregate benefit. The *potential value* of the market is the sum of the worths attached to the essential coalitions of any optimal assignment (or matching).

In these markets, knowing when agents are complements or substitutes is an important issue. For instance, a Government planning to promote the increase of the number of actors in some market should previously investigate if subsidies to achieve that goal must be given to firms of different sectors or to firms of the same sector and if these subsidies must be given at the same time to all the firms or not, facts that depend heavily on the complementarity and substitutability of firms and consumers in that market.

On one hand, in this paper I show that, given an arbitrary multi-sided assignment market with more than two sectors, agents of different types (resp. of the same type) need not be complements (resp. substitutes), in the sense that each of them need not (resp. can) reinforce each other so that the increase of the potential value of the market is larger (resp. lower) when they incorporate to the market jointly than when they do it separately. Results on complementarity and substitutability among agents are then proved to hold in arbitrary multi-sided assignment markets so that they imply Shapley's (1962) result. On the other hand, I show that, in markets where each agent can be assigned an individual productivity, in a sense that will be more concrete below, agents of different sectors are indeed complements and agents of the same sector are indeed substitutes. This type of m-sided markets embodies two-sided Böhm-Bawerk horse markets (Böhm-Bawerk, 1923 and Shapley and Subik, 1972) and, more generally, assignment markets with $m-1$ homogeneous goods (Tejada and Rafels, 2009).

Finally, the rest of the paper is organized as follows. In Section 2, I present the formal model and Shapley's (1962) result. Section 3 is devoted to present the main results of the paper regarding the complementarity and substitutability among agents for arbitrary multilateral assignment markets, whereas Section 4 is devoted to study the particular case of markets with homogeneous goods.

2 The model

Consider a market in which there are $m \geq 2$ types of agents, each of them containing a numerable infinite set of (potential) agents, being agent i of type j denoted by j-i.[1] Assume that there is also a mapping $A : \mathbb{N}^m \longrightarrow \mathbb{R}_+$, where $A(i_1, i_2, ..., i_m)$ stands for the worth created by the *essential coalition* $\{1\text{-}i_1, 2\text{-}i_2..., m\text{-}i_m\} \simeq (i_1, i_2, ..., i_m)$. Markets in which worth can only be created when different types of agents meet are found in many situations, for instance, an architect, a constructor and a notary are needed in a project to build a house. Let $N^1, ..., N^m$ be finite sets of agents of each type respectively picked from the potential sets of agents, where $\left|N^j\right| = n_j$ for all $1 \leq j \leq m$ and $N = N^1 \cup ... \cup N^m$. A m-sided assignment problem (shortly m-SAP) is defined by giving the sets $N^1, ..., N^m$ and the restriction of mapping A to N and it is denoted by $(N^1, ..., N^m; A_{|N})$ or simply $(N^1, ..., N^m; A)$ when no confusion can arise.

A *matching* $\mu = \{E^1, ..., E^t\}$ among $N^1, ..., N^m$ is a set of essential coalitions such that $|\mu| = t = \min_{1 \leq j \leq m} |N^j|$ and any agent j-i belongs at most to one essential coali-

[1] Formally, the i-th agent of type j should be denoted by a pair $(j, i) \in \{1, ..., m\} \times \mathbb{N}$.

tion $E^1, ..., E^t$. Agent j-i is *unassigned* by μ if she does not belong to E^k for any $1 \leq k \leq t$. The set of all matchings among $N^1, ..., N^m$ will denoted by $\mathcal{M}(N^1, ..., N^m)$. A matching μ^* is optimal if $\sum_{(i_1,...,i_m)\in\mu^*} A(i_1, ..., i_m) \geq \sum_{(i_1,...,i_m)\in\mu} A(i_1, ..., i_m)$, for any $\mu \in \mathcal{M}(N^1, ..., N^m)$. The set of all optimal matchings of $(N^1, ..., N^m; A)$ will be denoted by $\mathcal{M}^*_A(N^1, ..., N^m)$. Since $n_1, ..., n_m$ are finite, at least one optimal matching exists and, therefore, $\mathcal{M}^*_A(N^1, ..., N^m)$ is always nonempty.

Following Shapley (1962), let $\mathcal{S}(A) = \sum_{(i_1,...,i_m)\in\mu^*} A(i_1, ..., i_m)$ be the *score* (or potential value) associated to $(N^1, ..., N^m; A)$, where $\mu^* \in \mathcal{M}^*_A(N^1, ..., N^m)$. If $\min_{1\leq j\leq m}\{n_j\} = 0$, $S(A)$ is arbitrarily set to zero. When there is no possible confusion, I will just write $\mathcal{S}$ instead of $\mathcal{S}(A)$. Given a *status-quo* multi-sided assignment game $(N^1, ..., N^m; A)$ and j-$i \notin N$, let $\mathcal{S}^{j\text{-}i}$ denote the score of the m-SAP obtained from $(N^1, ..., N^m; A)$ by adding agent i of type j to N. More generally, given a m-SAP $(N^1, ..., N^m; A)$, which I name the *status quo* m-SAP, I shall use $S^{j_1\text{-}i_1,...,j_s\text{-}i_s}$ (resp. $S_{j_1\text{-}i_1,...,j_r\text{-}i_r}$) to denote the score associated to the corresponding m-SAP obtained from the status quo m-SAP by adding (resp. removing) agents j_1-$i_1, ..., j_s$-i_s to (resp. from) N. Since the exit or the arrival of new agents is symmetric, I will focus only on the latter. In such case I will always assume that j_1-$i_1, ..., j_s$-$i_s \notin N$ and, therefore, they will be referred to as *new agents*.

Shapley (1962) shows the following result.

Theorem 1 *Let $(N^1, N^2; A)$ be a status quo 2-SAP. Then,*

(a) $\left(\mathcal{S}^{j\text{-}i_j} - \mathcal{S}\right) + \left(\mathcal{S}^{k\text{-}l_k} - \mathcal{S}\right) \leq \mathcal{S}^{j\text{-}i_j,k\text{-}l_k} - \mathcal{S}$ *if* $1 \leq j < k \leq 2$,

(b) $\left(\mathcal{S}^{j\text{-}i_j} - \mathcal{S}\right) + \left(\mathcal{S}^{k\text{-}l_k} - \mathcal{S}\right) \geq \mathcal{S}^{j\text{-}i_j,k\text{-}l_k} - \mathcal{S}$ *if* $1 \leq j = k \leq 2$,

where j-i_j and k-l_k are new agents.

In words, Shapley shows that agents of different types are *complements*, since the increase on the potential value of the market is larger when two agents of different types incorporate to the market at the same time, $\mathcal{S}^{j\text{-}i_j,k\text{-}i_k} - \mathcal{S}$, than when they do it separately, $\left(\mathcal{S}^{j\text{-}i_j} - \mathcal{S}\right) + \left(\mathcal{S}^{k\text{-}i_k} - \mathcal{S}\right)$. By a similar argument, agents of the same type are *substitutes*.

Lastly, the marginal contribution of any agent $j\text{-}i$ in a market measures her impact in its potential value, formally $\mathcal{S}^{j\text{-}i} - \mathcal{S}$. Then observe that $\left(\mathcal{S}^{j\text{-}i_j} - \mathcal{S}\right) + \left(\mathcal{S}^{k\text{-}l_k} - \mathcal{S}\right) \leq \mathcal{S}^{j\text{-}i_j,k\text{-}l_k} - \mathcal{S}$ can be rearranged to $\mathcal{S}^{k\text{-}l_k} - \mathcal{S} \leq \left(\mathcal{S}^{j\text{-}i_j}\right)^{k\text{-}l_k} - \mathcal{S}^{j\text{-}i_j}$, which tells that the marginal contribution of agent $k\text{-}l_k$ cannot decrease when agent $j\text{-}i_j$ is in the market compared to when she is not.

3 Complements and substitutes in arbitrary multilateral assignment markets

The starting point of this section is Theorem 1, that holds for two-sided assignment problems. A natural question is to ask whether in assignment problems with more than two sides agents of the same type are substitutes and agents of different types are complements? The following example shows that the answer to both questions is negative, regardless the notion of complementarity considered.

Example 1 *Consider the 3-SAP defined by:*

$$\begin{array}{cccc} & i_2=1 \qquad i_2=2 & i_2=1 \qquad i_2=2 & i_2=1 \qquad i_2=2 \\ \begin{array}{c} i_1=1 \\ i_1=2 \end{array} & \begin{pmatrix} 1 & 7 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}. \\ & i_3=1 & i_3=2 & i_3=3 \end{array}$$

On one hand, let the status quo 3-SAP set of agents be $N = \{1\text{-}1, 2\text{-}1, 3\text{-}1\}$. It can be easily checked that $\mathcal{S} = 1$, $\mathcal{S}^{1\text{-}2} = 3$, $\mathcal{S}^{2\text{-}2} = 7$, $\mathcal{S}^{3\text{-}2} = 3$, $\mathcal{S}^{2\text{-}2,3\text{-}2} = 7$ and $\mathcal{S}^{1\text{-}2,2\text{-}2,3\text{-}2} = 7$. Hence,

$(\mathcal{S}^{2\text{-}2}-\mathcal{S})+(\mathcal{S}^{3\text{-}2}-\mathcal{S})=6+2>6=\mathcal{S}^{2\text{-}2,3\text{-}2}-\mathcal{S}$ and $(\mathcal{S}^{1\text{-}2}-\mathcal{S})+(\mathcal{S}^{2\text{-}2}-\mathcal{S})+(\mathcal{S}^{3\text{-}2}-\mathcal{S})=2+6+2>6=\mathcal{S}^{1\text{-}2,2\text{-}2,3\text{-}2}-\mathcal{S}$. In other words, neither agents 1-2, 2-2 and 3-2 all together nor agents 2-2 and 3-2 alone are complements. Observe that, for instance, what makes possible agents 2-2 and 3-2 not being complements is that the complementarity effect of introducing them jointly to the status quo m-SAP is diminished due to the existence of sector 1, that makes impossible to pick up simultaneously $A(1,2,1)=7$ and $A(1,1,2)=3$.

On the other hand, let the status quo 3-SAP set of agents be $N=\{1\text{-}1,1\text{-}2,2\text{-}1,2\text{-}2,3\text{-}1\}$. As above, it can be easily checked that $\mathcal{S}=4$, $\mathcal{S}^{3\text{-}2}=7$, $\mathcal{S}^{3\text{-}3}=10$ and $\mathcal{S}^{3\text{-}2,3\text{-}3}=13$, which imply that $(\mathcal{S}^{3\text{-}2}-\mathcal{S})+(\mathcal{S}^{3\text{-}3}-\mathcal{S})=0+3<6=\mathcal{S}^{3\text{-}2,3\text{-}3}-\mathcal{S}$. In other words, agents 3-2 and 3-3 are not substitutes although they belong to the same type. What makes this possible is that the substitutability effect of introducing agents 3-2 and 3-3 to the status quo m-SAP is offset by a complementarity effect between the same agents due to the existence of sector 1.

Next I show that, in spite of the above counterexamples, Theorem 1 can be generalized to include extended notions of complementarity and substitutability among groups of agents in arbitrary multi-sided assignment markets when these groups have, in some sense, bilateral features.

Nevertheless, before I do so, I introduce a definition that will be useful along the paper. Given $(N^1,...,N^m;A)$ an status quo m-SAP, let $1\text{-}i_1,...,t\text{-}i_t$ be some new agents, where $1\leq t\leq m-1$. These new agents are said to be *linked within* $(N^1,...,N^m;A)$ if $1\text{-}i_1,...,t\text{-}i_t$ are matched together or unmatched under some optimal matching of the m-SAP obtained from the status quo m-SAP by adding agents $1\text{-}i_1,...,t\text{-}i_m$. It is not difficult to check that in bilateral assignment problems ($m=2$ and $t=1$), agents are always linked.

Having introduced the notion of link, I am now in the position to give sufficient (but not necessary) conditions to guarantee that agents are complements in such a way that Part (a) of Theorem 1 is generalized.

Theorem 2 (Complementarity) *Let $(N^1,...,N^m;A)$ be a status quo m-SAP and let t be an integer such that $1\leq t\leq m-1$. If $1\text{-}i_1,...,m\text{-}i_m$ are new agents such that $1\text{-}i_1,...,t\text{-}i_t$ and*

$(t+1)$-$i_{t+1},...,m$-i_m *are respectively linked within* $(N^1,...,N^m;A)$*, then*

$$(1)\qquad \left(\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t}-\mathcal{S}\right)+\left(\mathcal{S}^{(t+1)\text{-}i_{t+1},...,m\text{-}i_m}-\mathcal{S}\right)\leq \mathcal{S}^{1\text{-}i_1,...,m\text{-}i_m}-\mathcal{S}.$$

Proof. The proof extends Shapley's (1962) one and consists on showing that (1) holds by induction on

$$(2)\qquad \lambda=\lambda(N^1,...,N^m)=\min_{1\leq j\leq m}|N^j|,$$

where $N^1,...,N^m$ are the set of agents of the status quo m-SAP.

On one hand, when $\lambda=0$, either $S^{1\text{-}i_1,...,m\text{-}i_m}=S=0$ or $S^{(t+1)\text{-}i_{t+1},...,m\text{-}i_m}=S=0$ and, therefore, (1) trivially holds. On the other hand, take $\lambda>0$ and suppose that, for any status quo m-SAP $(\widetilde{N}^1,...,\widetilde{N}^m;\widetilde{A})$ with $\widetilde{\lambda}=\lambda(\widetilde{N}^1,...,\widetilde{N}^m)$ such that $0\leq\widetilde{\lambda}\leq\lambda-1$, I have already proved that (1) holds. I will refer to this assumption as (IH). Then, consider $(N^1,...,N^m;A)$ a status quo m-SAP with $\lambda=\lambda(N^1,...,N^m)$ and suppose that (1) does not hold, i.e.

$$(3)\qquad \left(\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t}-\mathcal{S}\right)+\left(\mathcal{S}^{(t+1)\text{-}i_{t+1},...,m\text{-}i_m}-\mathcal{S}\right)> \mathcal{S}^{1\text{-}i_1,...,m\text{-}i_m}-\mathcal{S}.$$

We can assume that 1-$i_1,...,t$-i_t and $(t+1)$-$i_{t+1},...,m$-i_m are respectively matched under some optimal matching of the m-SAP obtained by adding the corresponding new agents to the status quo m-SAP, since, if any of these conditions is not satisfied, (1) trivially holds. Let $(i_1,...,i_t,k_{t+1},...,k_m)$ (resp. $(k_1,...,k_t,i_{t+1},...,i_m)$) denote the essential coalition that contains $\{1\text{-}i_1,...,t\text{-}i_t\}$ (resp. $\{(t+1)\text{-}i_{t+1},...,m\text{-}i_m\}$) and belongs to some optimal matching of the m-SAP obtained from the status quo m-SAP by adding 1-$i_1,...,t$-i_t (resp. $(t+1)$-$i_{t+1},...,m$-i_m). Since by hypothesis 1-$i_1,...,t$-i_t and $(t+1)$-$i_{t+1},...,m$-i_m are respectively linked within $(N^1,...,N^m;A)$,

$$\begin{aligned}
\mathcal{S}_{(t+1)\text{-}k_{t+1},...,m\text{-}k_m}+A(i_1,...,i_t,k_{t+1},...,k_m) &= \mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t}\\
\mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t}+A(k_1,...,k_t,i_{t+1},...,i_m) &= \mathcal{S}^{(t+1)\text{-}i_{t+1},...,m\text{-}i_m}\\
\mathcal{S}^{1\text{-}i_1,...,m\text{-}i_m} &\geq \mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\\
&\quad +A(i_1,...,i_t,k_{t+1},...,k_m)\\
&\quad +A(k_1,...,k_t,i_{t+1},...,i_m).
\end{aligned}$$

Finally, adding up all the above inequalities and (3),

$$
\begin{aligned}
& \left(\mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right) \\
& + \left(\mathcal{S}_{(t+1)\text{-}k_{t+1},...,m\text{-}k_m} - \mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right) \\
> \quad & \left(\mathcal{S} - \mathcal{S}_{1\text{-}k_1,...,t\text{-}k_t,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right),
\end{aligned}
$$

which contradicts the induction hypothesis (IH) because

$$
\begin{aligned}
\lambda' &= \lambda(N^1\backslash\{1\text{-}k_1\}, ..., N^1\backslash\{t\text{-}k_t\}, N^1\backslash\{(t+1)\,\text{-}k_{t+1}\}, ..., N^m\backslash\{m\text{-}k_m\}) \\
&= \min_{1\le j\le m} |N^j\backslash\{j\text{-}k_j| = \lambda(N^1, ..., N^m) - 1.
\end{aligned}
$$

■

Some remarks must be made on the above theorem. On one hand, let $\theta = (\theta(1), \theta(2), ..., \theta(m))$ be an ordering of the set of types $J = \{1, ..., m\}$, i.e. a bijective mapping from J to J. Observe that, in the above theorem and for simplicity, I have implicitly chosen the natural ordering of types of agents, $\theta = (1, 2, ..., m)$. However, since types can be relabeled for convenience, Theorem 2 can be formulated for any set of new agents $\{1\text{-}i_1, ..., m\text{-}i_m\}$ that is partitioned into $\left\{\theta(1)\text{-}i_{\theta(1)}, ..., \theta(t)\text{-}i_{\theta(t)}\right\}$ and $\left\{\theta(t+1)\text{-}i_{\theta(t+1)}, ..., \theta(m)\text{-}i_{\theta(m)}\right\}$, for any ordering θ and any integer $1 \le t \le m-1$.

Second, it is straightforward to check that Theorem 2 implies Part (a) of Theorem 1, since as I have already said, any agent is always linked within any status quo 2-sided assignment problem.

Third, if the link hypothesis of the above theorem is relaxed, complementarity among agents might not hold. To see it, take a look again at Example 1 and consider the status quo 3-SAP where $N = \{1\text{-}1, 2\text{-}1, 3\text{-}1\}$. Observe that 2-2 and 3-2 are not linked within the 3-SAP obtained from the status quo 3-SAP by adding agents 2-2, 3-2. It is also straightforward to check that $\left(\mathcal{S}^{1\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{2\text{-}2,3\text{-}2} - \mathcal{S}\right) = 2 + 6 > 6 = \mathcal{S}^{1\text{-}2,2\text{-}2,3\text{-}2} - \mathcal{S}$ and, hence, (1) does not hold.

Lastly, it is not difficult to find examples where the link hypothesis does not hold but (1) still does.

Now I turn into studying the substitutability among agents of the same type in arbitrary multi-sided assignment problems. I shall show that the link hypothesis is not sufficient to guarantee that agents of the same type are always substitutes. Instead, I find a more restrictive but sufficient condition. Specifically, I shall prove that agents are substitutes when 'they behave jointly as individuals', in a sense that I explain next. To do so, however, I previously need to extend the definition of link to support more general sets of new agents.

Indeed, suppose that $P^1 = (N^1, ..., N^m; A)$ is a status quo m-SAP and let $\{1\text{-}i_1, ..., t\text{-}i_t, 1\text{-}k_1, ..., t\text{-}k_t\}$ be a set of new agents, where $1 \leq t \leq m-1$ and, for all $1 \leq j \leq t$, $i_j \neq k_j$. With some formal abuse, these new agents are also said to be *linked within* $(N^1, ..., N^m; A)$ if $1\text{-}i_1, ..., t\text{-}i_t$ and $1\text{-}k_1, ..., t\text{-}k_t$ are respectively matched together or unmatched under some optimal matching μ of the m-SAP obtained from the status quo m-SAP by adding agents $1\text{-}i_1, ..., t\text{-}i_t, 1\text{-}k_1, ..., t\text{-}k_t$. Observe that two different agents for each type j, $1 \leq j \leq t$ and the same optimal matching for all of them are considered. Let $(i_1, ..., i_t, i_{t+1}, ..., i_m)$, $(k_1, ..., k_t, k_{t+1}, ..., k_m) \in N^1 \times ... \times N^m$ be the essential coalitions (if exist) that belong to μ and contain respectively $1\text{-}i_1, ..., t\text{-}i_t$ and $1\text{-}k_1, ..., t\text{-}k_t$.

Suppose that at least $1\text{-}i_1, ..., t\text{-}i_t$ are assigned together under μ. In a next step, let P^2 be the m-SAP obtained from P^1 by removing agents $(t+1)\text{-}i_{t+1}, ..., m\text{-}i_m$ and $(t+1)\text{-}k_{t+1}, ..., m\text{-}k_m$ (or just $(t+1)\text{-}i_1, ..., m\text{-}i_m$ if $1\text{-}k_1, ..., t\text{-}k_t$ were unmatched under μ). Observe that $(t+1)\text{-}i_{t+1}, ..., m\text{-}i_m$ and $(t+1)\text{-}k_{t+1}, ..., m\text{-}k_m$ (or just $(t+1)\text{-}i_1, ..., m\text{-}i_m$) are new agents to P^2. Therefore, $(t+1)\text{-}i_{t+1}, ..., m\text{-}i_m$ and $(t+1)\text{-}k_{t+1}, ..., m\text{-}k_m$ (or just $(t+1)\text{-}i_{t+1}, ..., m\text{-}i_m$) can be linked within P^2 or not. Suppose that they are and repeat the above procedure by removing the corresponding agents until either the corresponding new agents are all unmatched under all optimal matchings of the corresponding m-SAP or there are no more agents to be removed. Also suppose that, at any step s, the corresponding new agents are linked within the corresponding m-SAP P^s. In such case, the sets of new agents at any step of the above procedure -either if there are two agents for each type of agents or just one agent per type- are said to be *strongly linked within* $(N^1, ..., N^m; A)$. In particular, $\{1\text{-}i_1, ..., t\text{-}i_m, 1\text{-}k_1, ..., t\text{-}$

$k_m\}$ are strongly linked. The strong link condition consists on applying iteratively the link condition. Hence, the former condition implies the latter.

Although complicated to write, given an status quo m-SAP, some new agents being strongly linked within it essentially tells that the set of types can be partitioned into two subsets and that optimal matchings can be constructed from the empty set by adding appropriate blocks of agents of both subsets until an optimal matching of the status quo m-SAP is reached. In other words, the so obtained matching is, to some extent, 'bilateral'.

To a better comprehension of this definition consider a status quo two-sided assignment problem ($m = 2$) and two new agents ($t = 1$), one of each side. Observe that, at any step, since $t = 1$, any of these new agents is either unmatched or 'assigned to herself' and to an agent -let call her an *optimal partner*- of the other side under any optimal matching of the status quo 2-SAP. Hence, as we knew, new agents are always linked. Moreover, since $m-t = 1$, any optimal partner (if exist) is always also linked within the corresponding 2-SAP, and, therefore, also strongly linked.

Having introduced the notion of strong link, I am now in the position to give sufficient (but not necessary) conditions to guarantee that agents are substitutes in such a way that I generalize Part (b) of Theorem 1.

Theorem 3 (Substitutability) *Let $(N^1, ..., N^m; A)$ be a status quo m-SAP and t an integer such that $1 \leq t \leq m-1$. If $1\text{-}i_1, ..., t\text{-}i_t, 1\text{-}k_1, ..., t\text{-}k_t$ are new agents that are strongly linked within $(N^1, ..., N^m; A)$, then*

$$(4) \qquad \left(\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S}\right) \geq \mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S},$$

provided that, for all $1 \leq j \leq t$, $i_j \neq k_j$.

Proof. Again, the proof is very similar to Shapley's (1962) one. It consists on showing that (4) holds by induction on

$$(5) \qquad \lambda = \lambda(N^1, ..., N^m) = \min\left\{\min_{1\leq j\leq t} |N^j|, \min_{t+1\leq j\leq m} \left(|N^j| - 1\right)\right\},$$

where $N^1, ..., N^m$ are the set of agents of the status quo m-SAP. Observe that, given a status quo m-SAP, for any type of new agents λ takes the minimum of its cardinality, whereas for the rest of types λ takes the minimum of their cardinality minus 1. The reason to that will be apparent below.

Let μ be an optimal matching of the m-SAP obtained from the status quo m-SAP by adding agents $1\text{-}i_1, ..., t\text{-}i_t, 1\text{-}k_1, ..., t\text{-}k_t$. Before starting the induction observe that if $1\text{-}k_1, ..., t\text{-}k_t$ are unmatched under μ, then $\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t} = \mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t}$, which trivially implies that (4) holds. Hence, from now on I shall assume that $1\text{-}i_1, ..., t\text{-}i_t$ and $1\text{-}k_1, ..., t\text{-}k_t$ are respectively matched together under μ. Therefore,

$$\{(i_1, ..., i_t, i_{t+1}, ..., i_m), (k_1, ..., k_t, k_{t+1}, ..., k_m)\} \subseteq \mu, \tag{6}$$

for some $(i_{t+1}, ..., i_m)$, $(k_{t+1}, ..., k_m) \in N^{t+1} \times ... \times N^m$ satisfying that, for all $t + 1 \leq j \leq m$, $i_j \neq k_j$.

Next I prove that (4) holds when $\lambda = 0$, which implies $\mathcal{S} = 0$. Observe that, necessarily, $\min_{t+1\leq j\leq m} \left\{|N^j| - 1\right\} > 0$, for if not either $1\text{-}i_1, ..., t\text{-}i_t$ or $1\text{-}k_1, ..., t\text{-}k_t$ would be unassigned under μ, contradicting the assumptions. Therefore, μ must consist exactly on two essential coalition, i.e. $|\mu| = 2$. Lastly, from (6), $\mathcal{S} = 0$ and the definition the score of a m-SAP,

$$\begin{aligned}\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S} &= A(i_1, ..., i_t, i_{t+1}, ..., i_m) + A(k_1, ..., k_t, k_{t+1}, ..., k_m) \\ &\leq \left(\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S}\right).\end{aligned}$$

On the other hand, take $\lambda > 0$ and suppose that, for any status quo m-SAP $(\widetilde{N}^1, ..., \widetilde{N}^m; \widetilde{A})$ with $\widetilde{\lambda} = \lambda(\widetilde{N}^1, ..., \widetilde{N}^m)$ such that $0 \leq \widetilde{\lambda} \leq \lambda - 1$, I have already proved that (4) holds for any set of new agents that are strongly linked within $(\widetilde{N}^1, ..., \widetilde{N}^m; \widetilde{A})$. I will refer to this assumption as (IH). Then, consider $(N^1, ..., N^m; A)$ a status quo m-SAP with $\lambda(N^1, ..., N^m) = \lambda$ and suppose that (4) does not hold for some new agents $1\text{-}i_1, ..., t\text{-}i_m, 1\text{-}k_1, ..., t\text{-}k_m$ that are strongly linked within $(N^1, ..., N^m; A)$. That is,

$$\left(\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S}\right) < \mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t} - \mathcal{S}. \tag{7}$$

Additionally, from (6), the strong link hypothesis and the definition the score of a m-SAP,

$$
\begin{aligned}
A(i_1,...,i_t,i_{t+1},...,i_m) + \mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,} &\leq \mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t} \\
A(k_1,...,k_t,k_{t+1},...,k_m) + \mathcal{S}_{(t+1)\text{-}k_{t+1},...,m\text{-}k_m} &\leq \mathcal{S}^{1\text{-}k_1,...,t\text{-}k_t} \\
\mathcal{S}^{1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t} &= \mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m} \\
&\quad + A(i_1,...,i_t,i_{t+1},...,i_m) \\
&\quad + A(k_1,...,k_t,k_{t+1},...,k_m)
\end{aligned}
$$

Finally, adding up all the above inequalities and (7),

$$
\begin{aligned}
&\left(\mathcal{S}_{(t+1)\text{-}k_{t+1},...,m\text{-}k_m} - \mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right) \\
&+ \left(\mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,} - \mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right) \\
> \quad &\left(\mathcal{S} - \mathcal{S}_{(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m}\right),
\end{aligned}
$$

which contradicts the induction hypothesis (IH) because

$$
\begin{aligned}
\lambda' &= \lambda(N^1,...,N^t,N^{t+1}\backslash\{(t+1)\text{-}i_{t+1},(t+1)\text{-}k_{t+1}\},...,N^m\backslash\{m\text{-}i_m,m\text{-}k_m\}) \\
&= \min\left\{\min_{t+1\leq j\leq m}\left(|N^j|-2\right), \min_{1\leq j\leq t}\left(|N^j|-1\right)\right\} \\
&= \min\left\{\min_{1\leq j\leq t}|N^j|, \min_{t+1\leq j\leq m}\left(|N^j|-1\right)\right\} - 1 = \lambda - 1
\end{aligned}
$$

and, since $1\text{-}i_1,...,t\text{-}i_t,1\text{-}k_1,...,t\text{-}k_t$ are strongly linked within $(N^1,...,N^m;A)$, their corresponding optimal partners $(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m$ are also strongly linked within the m-SAP obtained from $(N^1,...,N^m;A)$ by removing agents $(t+1)\text{-}i_{t+1},...,m\text{-}i_m,(t+1)\text{-}k_{t+1},...,m\text{-}k_m$. ■

Again, some remarks must be made. On one hand, as above, Theorem 3 can be formulated for any arbitrary ordering of types θ.

Second, Theorem 3 implies part (b) of Theorem 1, since any agent is always strongly linked within any status quo 2-sided assignment problem.

Third, if the strong link hypothesis of the above theorem is dropped, the result might not hold. To see it, take a look again at Example 1 and consider the status quo 3-SAP where

$N = \{1\text{-}1, 1\text{-}2, 2\text{-}1, 2\text{-}2, 3\text{-}1\}$. I have already shown that, in such case, $\left(\mathcal{S}^{3\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{3\text{-}3} - \mathcal{S}\right) < \mathcal{S}^{3\text{-}2,3\text{-}3} - \mathcal{S}$. I claim that 3-2 and 3-3 are not strongly linked within the status quo 3-SAP, although, by definition, they are linked within it. To show it, notice that the optimal partners of 3-2 and 3-3 are respectively 1-1, 2-1 and 1-2, 2-2. Then observe that 1-1, 2-1 and 1-2, 2-2 are not respectively assigned together under the (unique) optimal matching of the status quo 3-SAP. Hence, 3-2 and 3-3 are not strongly linked within the status quo 3-SAP.

Fourth, like in the case of Theorem 2, it is not difficult to find examples where the strong link hypothesis does not hold but (4) does.

Finally, I recognize that the strong link condition might seem a little bit ad-hoc. However, it shows that substitution among agents is, to some extent, a bilateral concept, in the sense that the substitution effect between groups of agents can always be offset when there are more than two groups of agents, unless markets are endowed with more structure -as it is shown below-. Moreover, it is also worth noting that an accurate review of the proof of Theorem 3 reveals that formulating its statement for an arbitrary t, $1 \leq t \leq m-1$ instead of just $t = 1$ -which is the most natural generalization of Shapley's (1962) substitution effect- is made at zero cost.

4 The particular case of multilateral assignment markets with homogeneous goods

The remaining part of the paper is devoted to show that, when each agent in the market can be assigned a 'productivity', in a sense that will be concrete in the next definition, agents of different sectors of the market are complements whereas agents of the same sector are substitutes, in the sense of Shapley (1962). Formally, I introduce the following definition (Tejada and Rafels, 2009).

Definition 1 *Given the sets of agents $N^1, ..., N^m$ and exogenous vectors $d_1 \in \mathbb{R}^{n_1}, ..., d_m \in \mathbb{R}^{n_m}$, the multi-sided Böhm-Bawerk assignment problem associated to $d = (d_1, ..., d_m)$ is the* *m-SAP* $(N^1, ..., N^m; A^d)$*, where*

$$A^d(i_1, ..., i_m) = \max\left\{0, \sum_{j=1}^{m} d_{ji_j}\right\}, \tag{8}$$

for any $(i_1, ..., i_m) \in N^1 \times ... \times N^m$.

We assume that within the universe of potential agents, each one of them, let us say j-i, is characterized by an individual productivity d_{ji}, which implies that regardless the agents picked up to form a market, it will always be a multi-sided Böhm-Bawerk market. This class of m-SAPs embodies assignment markets with $m-1$ types homogeneous goods and in the bilateral case coincide with the well-know (two-sided) Böhm-Bawerk horse market (see again Tejada and Rafels, 2009). Without loss of generality I will assume that, for all $1 \leq j \leq m$, $d_{j1} \geq ... \geq d_{jn_j}$. Let $n = \min\{n_1, ..., n_m\}$. An important parameter is defined:

$$r = \max_{1 \leq i \leq n} \left\{i : A^d(i, ..., i) > 0\right\}. \tag{9}$$

Since it can be easily checked that $\{(i, ..., i) : 1 \leq i \leq n\}$ is an optimal matching of $(N^1, ..., N^m; A^d)$, from (9), we have

$$S(A^d) = \sum_{i=1}^{r} A^d(i, ..., i) = \sum_{i=1}^{r} \sum_{j=1}^{m} d_{ji}. \tag{10}$$

Next I prove that, for Böhm-Bawerk assignment problems, agents of different sectors are complements in a very strong form, although they might not be linked within the status quo Böhm-Bawerk assignment problem, whereas different agents of the same sector are substitutes regardless being strongly linked within the status quo m-sided Böhm-Bawerk assignment problem or not. Nevertheless, before I do so it is important to point out that, unlike Theorem 2 and Theorem 3, the following theorem does not provide conditions on arbitrary essential coalitions of arbitrary m-SAPs but only focuses on a specific class of m-sided assignment markets.

Theorem 4 *Let* $(N^1, ..., N^m; A^d)$ *be a m-sided Böhm-Bawerk assignment problem. Then,* *(a) for any essential coalition of new agents $(i_1,...,i_m)$ and any $1=j_0<j_1<...<j_s<j_{s+1}=m+1$,*

$$\sum_{t=0}^{s}\left(\mathcal{S}^{j_t\text{-}i_{j_t},...,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}}-S\right)\leq\mathcal{S}^{1\text{-}i_1,...,m\text{-}i_m}-\mathcal{S},\tag{11}$$

(b) for any pair of different new agents j-i, j-k and any $1\leq j\leq m$,

$$\left(\mathcal{S}^{j\text{-}i}-\mathcal{S}\right)+\left(\mathcal{S}^{j\text{-}k}-\mathcal{S}\right)\geq\mathcal{S}^{j\text{-}i,j\text{-}k}-\mathcal{S}.\tag{12}$$

Proof. Without loss of generality, I shall assume throughout the whole proof, that the status quo m-sided Bhm-Bawerk assignment problem $(N^1,...,N^m;A^d)$ satisfies $r<n$.[2] Additionally, let $r^{1\text{-}k_1,...,t\text{-}k_t}$ denote the parameter (9) of the m-SAP obtained by adding agents 1-$k_1,...,t$-k_t to the status quo m-SAP, for any $0\leq t\leq m$. By definition of (9),

$$r\leq r^{1\text{-}k_1,...,t\text{-}k_t}\leq r+1.\tag{13}$$

Then, I prove Part (a). To do so, consider, for all $t\in\{1,...,s+1\}$, a set of new agents 1-$i_1^t,...,m$-i_m^t with productivities $d_{ji_j^t}=d_{ji_j}$, for all $1\leq j<j_t$, and $d_{ji_j^t}=d_{j(r+1)}$, for all $j_t\leq j\leq m$.

Next, I shall show that (11) holds when the set of new agents is $\{1\text{-}i_1^1,...,m\text{-}i_m^1\}$. Indeed, observe that, for all $j_1\leq j\leq m$, j-i_j^1 and j-$(r+1)$ are different agents of the same side with the same productivities. Therefore, when agents 1-$i_1^1,...,(j_1-1)$-$i_{j_1-1}^1$ incorporate into the market, the increase of the score is the same irrespective agents j_1-$i_{j_1}^1,...,m$-i_m^1 also incorporate into the market or not. Moreover, for any $0\leq t\leq m$, the impact on the score when only agents of types $j\in\{j_t,...,m\}$ incorporate into the market is always null. All the above is true by (13) and because, if needed, agents j_1-$(r+1)$ $,...,m$-$(r+1)$ of the status quo

[2] This can be made by adding dummy agents. A dummy agent is such that the worth of any essential coalition 'containing' her is always zero. Dummy agents do not alter the characteristic function when we restrict to real players (see Tejada and Rafels, 2009, for a more accurate argument). In m-sided Bhm-Bawerk assignment problems, dummy agents can be imagined to have a very low productivity -think about minus infinity.

m-SAP can play the role of agents j_1-$i^1_{j_1}, ..., m$-i^1_m -they have the same productivities!- and, by (10), they do not count on the score of the status quo m-SAP. Hence,

$$\begin{aligned} & \left(\mathcal{S}^{1\text{-}i^1_1,...,(j_1-1)\text{-}i^1_{j_1-1}} - S\right) + \sum_{l=2}^{s}\left(\mathcal{S}^{j_l\text{-}i_{j_1},...,(j_{l+1}-1)\text{-}i_{(j_{l+1}-1)}} - S\right) \\ = \; & \mathcal{S}^{1\text{-}i^1_1,...,(j_1-1)\text{-}i^1_{j_1-1}} - S = \mathcal{S}^{1\text{-}i^1_1,...,m\text{-}i^1_m} - \mathcal{S}, \end{aligned}$$

and therefore (11) holds for 1-$i^1_1, ..., m$-i^1_m. Now suppose that (11) holds when the set of new agents is $\{1\text{-}i^t_1, ..., m\text{-}i^t_m\}$, for some $1 \leq t \leq s-1$. Next, I will show that it also holds for $t+1$. Indeed, observe that only agents of types $j_t, ..., j_{t+1}-1$ can be different in the sets of new agents $\{1\text{-}i^{t+1}_1, ..., m\text{-}i^{t+1}_m\}$ and $\{1\text{-}i^t_1, ..., m\text{-}i^t_m\}$. Hence, since for all $1 \leq t \leq t'$ and for all $1 \leq j < k_t$, we have that $d_{ji^t_j} = d_{ji^{t'}_j}$, whereas for all $1 \leq t' \leq t$ and for all $k_t \leq j \leq m$ we have that $d_{ji^{t'}_j} = d_{j(r+1)}$, then

$$\begin{aligned} & \sum_{l=0}^{s}\left(\mathcal{S}^{j_l\text{-}i^{t+1}_{j_l},...,(j_{l+1}-1)\text{-}i^{t+1}_{(j_{l+1}-1)}} - S\right) \\ = \; & \sum_{t=0}^{s}\left(\mathcal{S}^{j_l\text{-}i^t_{j_l},...,(j_{l+1}-1)\text{-}i^t_{(j_{l+1}-1)}} - S\right) \\ & + \mathcal{S}^{j_t\text{-}i^{t+1}_{j_t},...,(j_{t+1}-1)\text{-}i^{t+1}_{(j_{t+1}-1)}} - \mathcal{S}^{j_t\text{-}i^t_{j_t},...,(j_{t+1}-1)\text{-}i^t_{(j_{t+1}-1)}} \\ \leq \; & \mathcal{S}^{1\text{-}i^t_1,...,m\text{-}i^t_m} - \mathcal{S} + \mathcal{S}^{j_t\text{-}i^{t+1}_{j_t},...,(j_{t+1}-1)\text{-}i^{t+1}_{(j_{t+1}-1)}} - \mathcal{S}^{j_t\text{-}i^t_{j_t},...,(j_{t+1}-1)\text{-}i^t_{(j_{t+1}-1)}} \\ \leq \; & \mathcal{S}^{1\text{-}i^t_1,...,m\text{-}i^t_m} - \mathcal{S} \\ & + \mathcal{S}^{1\text{-}i^t_1,...,(j_t-1)\text{-}i^t_{(j_t-1)},j_t\text{-}i^{t+1}_{j_t},...,(j_{t+1}-1)\text{-}i^{t+1}_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i^t_1,...,(j_t-1)\text{-}i^t_{(j_t-1)}} \\ = \; & \mathcal{S}^{1\text{-}i^{t+1}_1,...,m\text{-}i^{t+1}_m} - \mathcal{S}, \end{aligned}$$

where the first inequality holds by the induction assumption and all the comments above. It only remains to prove the last inequality, which using the above comments, reduces to

$$\mathcal{S}^{j_t\text{-}i_{j_t},...,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S} \leq \mathcal{S}^{1\text{-}i_1,...,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i_1,...,(j_t-1)\text{-}i_{(j_t-1)}}. \tag{14}$$

Notice that, by (9) and (10), for any l-$k_l, ..., h$-k_h, $1 \leq l \leq h \leq m$, either

$$\mathcal{S}^{1\text{-}k_1,...,h\text{-}k_h} - \mathcal{S} = \sum_{j=l}^{h} \max\left\{0, d_{jk_j} - d_{jr}\right\}$$

if $r^{l\text{-}k_l,\ldots,h\text{-}k_h} = r$ or

$$\mathcal{S}^{1\text{-}k_1,\ldots,h\text{-}k_h} - \mathcal{S} = \sum_{j=1}^{l-1} d_{j(r+1)} + \sum_{j=l}^{h} \max\left\{d_{jk_j}, d_{j(r+1)}\right\} + \sum_{j=h+1}^{m} d_{j(r+1)}$$

if $r^{l\text{-}k_l,\ldots,h\text{-}k_h} = r+1$.

To prove (14) I shall distinguish four cases. First, suppose that

$$\begin{aligned}
(15) \qquad r &= r^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}}, \\
(16) \qquad r &= r^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}} < r^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} = r+1.
\end{aligned}$$

For all $1 \le j \le m$, let

$$h_j = d_{j(r+1)} + \max\{0, d_{ji_j} - d_{j(r+1)}\} - \max\{0, d_{ji_j} - d_{jr}\}.$$

and observe that, by (9) and (10), the right-hand side of (16) implies $\sum_{j=1}^{j_{t+1}-1} h_j + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} > 0$. Then,

$$\begin{aligned}
\mathcal{S}^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S} &= \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - d_{jr}\} \\
&< \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - d_{jr}\} + \sum_{j=1}^{j_{t+1}-1} h_j + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} \\
&= \sum_{j=1}^{j_t-1} h_j + \sum_{j=j_t}^{j_{t+1}-1} \max\{d_{ji_j}, d_{j(r+1)}\} + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} \\
&= \mathcal{S}^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}}.
\end{aligned}$$

Second, suppose that

$$\begin{aligned}
r &< r^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} = r+1, \\
r^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}} &= r^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} = r+1.
\end{aligned}$$

By (9), $\sum_{j=1}^{m} d_{j(r+1)} \leq 0$. Then,

$$\begin{aligned}
\mathcal{S}^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S} &= \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - d_{j(r+1)}\} \\
&\geq \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - d_{j(r+1)}\} + \sum_{j=1}^{m} d_{j(r+1)} \\
&= \sum_{j=1}^{j_t-1} d_{j(r+1)} + \sum_{j=j_t}^{j_{t+1}-1} \max\{d_{ji_j}, d_{j(r+1)}\} + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} \\
&= \mathcal{S}^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}}.
\end{aligned}$$

Third, suppose that

$$r = r^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}}, \tag{17}$$

$$r^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}} = r^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}}. \tag{18}$$

For all $1 \leq j \leq m$, let $h'_j = d_{jr}$ if (18) is equal to r and $h'_j = d_{j(r+1)}$ if (18) is equal to $r+1$.

Then,

$$\begin{aligned}
\mathcal{S}^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S} &= \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - d_{jr}\} \\
&\leq \sum_{j=j_t}^{j_{t+1}-1} \max\{0, d_{ji_j} - h'_j\} \\
&= \mathcal{S}^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}}.
\end{aligned}$$

Fourth, suppose that

$$\begin{aligned}
r &< r^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} = r+1, \\
r^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}} &< r^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} = r+1.
\end{aligned}$$

In such case,

$$\begin{aligned}
&\mathcal{S}^{1\text{-}i_1,\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}^{1\text{-}i_1,\ldots,(j_t-1)\text{-}i_{(j_t-1)}} \\
&= \sum_{j=1}^{j_t-1} h_j + \sum_{j=j_t}^{j_{t+1}-1} \max\{d_{ji_j}, d_{j(r+1)}\} + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} \\
&\geq \sum_{j=1}^{j_t-1} d_{j(r+1)} + \sum_{j=j_t}^{j_{t+1}-1} \max\{d_{ji_j}, d_{j(r+1)}\} + \sum_{j=j_{t+1}}^{m} d_{j(r+1)} \\
&= \mathcal{S}^{j_t\text{-}i_{j_t},\ldots,(j_{t+1}-1)\text{-}i_{(j_{t+1}-1)}} - \mathcal{S}.
\end{aligned}$$

Lastly, applying recursively the above procedure from $t = 1$ to $t = s$ it is proved that (11) holds for $\{1\text{-}i_1, ..., m\text{-}i_m\}$.

Now I prove Part (b). Without loss of generality assume that $j = 1$. Observe that, by (9),

$$r^{1\text{-}i,1\text{-}k} - r \leq r^{1\text{-}i} - r + r^{1\text{-}k} - r \leq 2 \tag{19}$$

and

$$\max\{r^{1\text{-}i}, r^{1\text{-}k}\} \leq r^{1\text{-}i,1\text{-}k}. \tag{20}$$

Next, I distinguish four different cases. First, suppose that $r^{1\text{-}i,1\text{-}k} = r+2$, which implies that $n_2, ..., n_m \geq r+2$ and that, by (19), $r^{1\text{-}i} = r^{1\text{-}k} = r+1$. Then, by the way d is arranged,

$$\begin{aligned} & \mathcal{S}^{1\text{-}i,1\text{-}k} - \mathcal{S} = d_{1i} + d_{1k} + \sum_{j=2}^{m} \left(d_{j(r+1)} + d_{j(r+2)}\right) \\ \leq \;& d_{1i} + \sum_{j=2}^{m} d_{j(r+1)} + d_{1k} + \sum_{j=2}^{m} d_{j(r+1)} = \left(\mathcal{S}^{1\text{-}i} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k} - \mathcal{S}\right). \end{aligned}$$

Second, suppose that $r^{1\text{-}i,1\text{-}k} = r^{1\text{-}i} = r^{1\text{-}k} = r+1$, which, by (9), implies that $d_{1i} + \sum_{j=2}^{m} d_{j(r+1)} > 0$, $d_{1k} + \sum_{j=2}^{m} d_{j(r+1)} > 0$ and $d_{1r} + \sum_{j=2}^{m} d_{j(r+1)} > 0$. Then,

$$\begin{aligned} \mathcal{S}^{1\text{-}i,1\text{-}k} - \mathcal{S} &= \max\{d_{1i}, d_{1k}, d_{1i} + d_{1k} - d_{1r}\} + \sum_{j=2}^{m} d_{j(r+1)} \\ &< d_{1i} + \sum_{j=2}^{m} d_{j(r+1)} + d_{1k} + \sum_{j=2}^{m} d_{j(r+1)} \\ &= \left(\mathcal{S}^{1\text{-}i} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k} - \mathcal{S}\right). \end{aligned}$$

Third suppose again that $r^{1\text{-}i,1\text{-}k} = r+1$ but, without loss of generality, that $r^{1\text{-}i} = r+1$ and $r^{1\text{-}k} = r$, which implies $d_{1i} > d_{1k}$. Then,

$$\begin{aligned} \mathcal{S}^{1\text{-}i,1\text{-}k} - \mathcal{S} &= \max\{d_{1i}, d_{1i} + d_{1k} - d_{1r}\} + \sum_{j=2}^{m} d_{j(r+1)} \\ &= d_{1i} + \max\{0, d_{1k} - d_{1r}\} + \sum_{j=2}^{m} d_{j(r+1)} \\ &= \left(\mathcal{S}^{1\text{-}i} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k} - \mathcal{S}\right). \end{aligned}$$

Finally, let $r^{1\text{-}i,1\text{-}k} = r$, which, by (20), implies $r^{1\text{-}i} = r^{1\text{-}k} = r$. Then, by the way d_1 is arranged,

$$\begin{aligned}
\mathcal{S}^{1\text{-}i,1\text{-}k} - \mathcal{S} &= \max\{0, d_{1i} - d_{1r}, d_{1k} - d_{1r}, d_{1i} + d_{1k} - d_{1r} - d_{1(r-1)}\} \\
&\leq \max\{0, d_{1i} - d_{1r}\} + \max\{0, d_{1k} - d_{1r}\} \\
&= \left(\mathcal{S}^{1\text{-}i} - \mathcal{S}\right) + \left(\mathcal{S}^{1\text{-}k} - \mathcal{S}\right).
\end{aligned}$$

■

On one hand, like Theorem 2 and Theorem 3, Part (a) of Theorem 4 can be formulated for any ordering of types.

On the other hand, by taking status quo 3-sided Böhm-Bawerk assignment problem generated by $d = (2; 2; 2, 1, 1)$, the reader can check that when agents $1\text{-}i_1, 2\text{-}i_2$ and $1\text{-}k_1, 2\text{-}k_2$, with productivities $1, -10$ and $-10, 1$ respectively, are new agents that incorporate to the status quo, (4) does not hold. Hence, the stronger form of substitutability of Theorem 3 does not extend to multilateral markets with homogeneous goods.

Finally, it is natural to wonder whether the class of m-sided Böhm-Bawerk assignment problems can be enlarged so that Theorem 4 still holds. A m-SAP $(N^1, ..., N^m; A)$ is *supermodular* (Sherstyuk, 1999) if

$$\begin{aligned}
& A(i_1, ..., i_m) + A(k_1, ..., k_m) \\
\leq \;& A(\max\{i_1, k_1\}, ..., \max\{i_m, k_m\}) + A(\min\{i_1, k_1\}, ..., \min\{i_m, k_m\}).
\end{aligned}$$

It is known that m-sided Böhm-Bawerk assignment problem are supermodular (Tejada and Rafels, 2009). Nevertheless, Theorem 4 fails to hold for some supermodular m-SAPs as I show next. First, consider the supermodular 3-SAP given by

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix},$$

where $N = \{1\text{-}2, 2\text{-}1, 3\text{-}1\}$ is the status quo. In this case, $\left(\mathcal{S}^{2\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{3\text{-}2} - \mathcal{S}\right) > \mathcal{S}^{2\text{-}2,3\text{-}2} - \mathcal{S}$ and $\left(\mathcal{S}^{1\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{2\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{3\text{-}2} - \mathcal{S}\right) > \mathcal{S}^{1\text{-}2,2\text{-}2,3\text{-}2} - \mathcal{S}$. Second, consider the supermodular 3-SAP given by

$$\begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}.$$

where $N = \{1\text{-}1, 1\text{-}2, 2\text{-}1, 2\text{-}2, 3\text{-}2\}$ is the status quo. In this case, $\left(\mathcal{S}^{3\text{-}2} - \mathcal{S}\right) + \left(\mathcal{S}^{3\text{-}3} - \mathcal{S}\right) = 2 < 4 = \mathcal{S}^{3\text{-}2,3\text{-}3} - \mathcal{S}$.

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